NORTH SYDNEY BOYS' HIGH SCHOOL
2008 Trial HSC Examination

MATHEMATICS

General instructions

- Working time – 3 hours.
  (plus 5 minutes reading time)
- Write on the lined paper in the booklet provided.
- **Each question is to commence on a new page.**
- Write using blue or black pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- **This paper must NOT be removed from the examination room by the candidate.**

Class teacher (please ✔)

◯ Mr Ireland

◯ Mr Lam

◯ Mr Lowe

◯ Mr Fletcher

◯ Mr Trenwith/Mr Taylor

STUDENT NUMBER: ....

Marker's use only.

QUESTION	1	2	3	4	5	6	7	8	9	10	Total	%
MARKS	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{12}$	$\overline{120}$	

Question 1 (12 Marks) Commence a NEW page. **Marks**

(a) Evaluate $\left(\frac{1}{e^{2.5}} - 1\right)^2$ correct to 3 significant figures. **2**

(b) Express $\frac{\sqrt{2}}{1+\sqrt{5}}$ with a rational denominator. **2**

(c) Differentiate $y = (4x+1)^3$ with respect to x. **2**

(d) Factorise $x^4y - xy^4$ fully. **2**

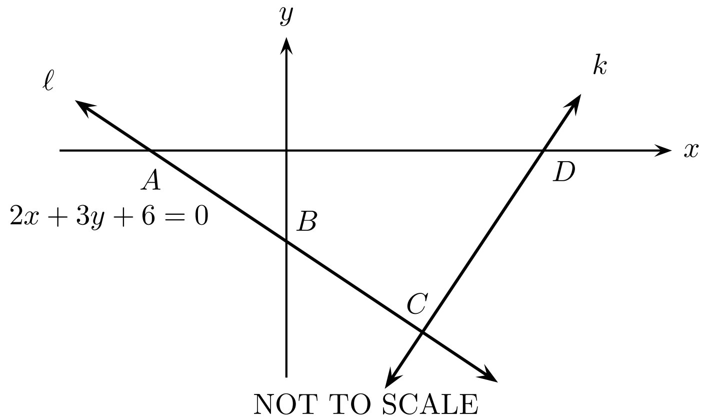
(e) Solve the following for x:

i. $2^{2x-3} = 32$. **2**

ii. $x^2 - x = 2$. **2**

Question 2 (12 Marks) Commence a NEW page. **Marks**

(a) The line ℓ has the equation $2x + 3y + 6 = 0$. It cuts the x axis at A and the y axis at B and it intersects the line k at C. Line k is perpendicular to ℓ and cuts the x axis at D.



Copy or trace the diagram on to your paper.

i. Find the coordinates of A. **1**

ii. Find the coordinates of B. **1**

iii. If B is the midpoint of AC prove that the coordinates of C are $(3, -4)$. **2**

iv. Show that the equation of k is given by $3x - 2y - 17 = 0$. **2**

v. Write the 3 inequalities required to define the interior region of $\triangle ACD$. **3**

(b) Find the equation of the tangent to the curve $y = x^2 \ln x$ at the point P where $x = e$. **3**

Question 3 (12 Marks) Commence a NEW page. **Marks**

(a) Consider the parabola $(x-2)^2 = 8(y+1)$.

i. Write down the focal length. **1**

ii. Write down the coordinates of the focus. **1**

iii. Find the equation of the directrix. **1**

(b) Differentiate with respect to x:

i. $2x^3 - x^{-1}$. **2**

ii. $\dfrac{\sin x}{e^{2x}}$. **2**

(c) Evaluate $\displaystyle\int_1^e \left(x^2 + \frac{2}{x}\right) dx$. **2**

(d) Find an approximation for $\displaystyle\int_1^3 g(x)\, dx$ by using Simpson's Rule with the following function values in the table below, correct to 2 decimal places. **3**

x	1	1.5	2	2.5	3
$g(x)$	12	8	0	3	5

Question 4 (12 Marks) Commence a NEW page. **Marks**

(a) Michael is training for a local marathon. He has trained by completing practice runs over the marathon course. So far he has completed three practice runs with times shown below.

Week	Time (hours)
1	3
2	2.7
3	2.43

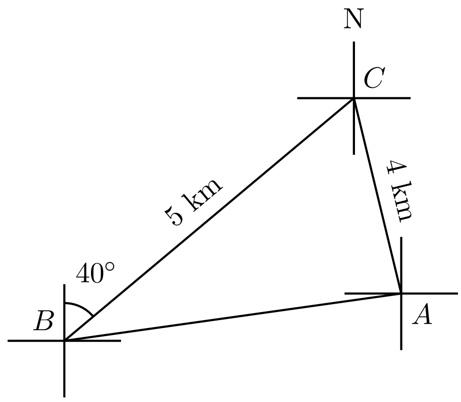
i. Show that these times form a geometric series with common ratio $r = 0.9$. **1**

ii. If this series continues, what would be his expected time in Week 5, completed to the nearest minute? **2**

iii. How many hours and minutes will he have run in total in his practice runs in these 5 weeks? **2**

iv. If the previous winning time for the marathon was 1 hour 15 min, how many weeks must he keep practising to be able to run the marathon in less that the previous winning time? **2**

(b) A, B and C are markers in an orienteering course. $AC = 4$ km and $BC = 5$ km. The bearing of C from B is 040°T.



Copy or trace the diagram into your writing booklet.

i. If the bearing of B from A is 260°T, show that $\angle CBA = 40°$, giving reasons. **2**

ii. Find $\angle CAB$ to the nearest degree. **2**

iii. Hence or otherwise, find the bearing of C from A. **1**

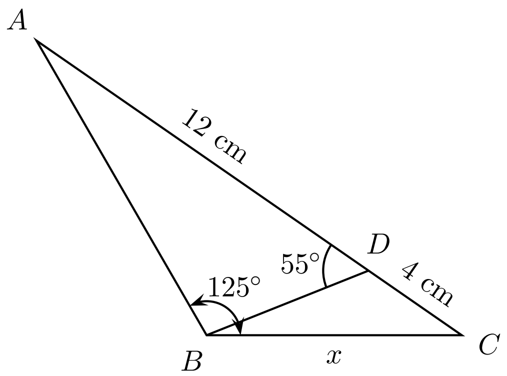
Question 5 (12 Marks) Commence a NEW page. **Marks**

(a) If $f(x) = x^2 - x$,

i. Evaluate and expand $f(x+h)$. **1**

ii. Hence or otherwise, differentiate $f(x) = x^2 - x$ from first principles. **3**

(b) In $\triangle ABC$ as shown in the diagram, $\angle ABC = 125°$, $\angle ADB = 55°$, $AD = 12$ cm and $DC = 4$ cm.



NOT TO SCALE

i. Show that $\triangle ABC \parallel\!| \triangle BDC$. **3**

ii. Find x, the length of BC. **2**

(c) Let α and β be the solutions of $2x^2 - 6x - 1 = 0$.

i. Find $\alpha + \beta$. **1**

ii. Find $\alpha\beta$. **1**

iii. Hence, find $3\alpha - \alpha^2$. **1**

Question 6 (12 Marks) Commence a NEW page. **Marks**

(a) A particle is moving in a straight line. Its velocity for $t \geq 0$ is given by

$$v = \frac{4}{t+1} - 2t$$

i. Find when the particle changes direction. **2**

ii. Find the exact distance travelled in the first 2 seconds. **3**

(b) For the function $y = x^3 - 3x^2 - 9x + 1$,

i. Find the coordinates of any stationary points and determine their nature. **3**

ii. Find any points of inflexion. **2**

iii. Neatly sketch the curve. **2**

Question 7 (12 Marks) Commence a NEW page. **Marks**

(a) A farmer has a large tank full of water. The tank leaks water from a hole. The volume of water remaining in the tank, in litres, is given by

$$V = 4\,000 + 10\,000e^{-0.04t}$$

where t is the time in hours after the leakage commenced.

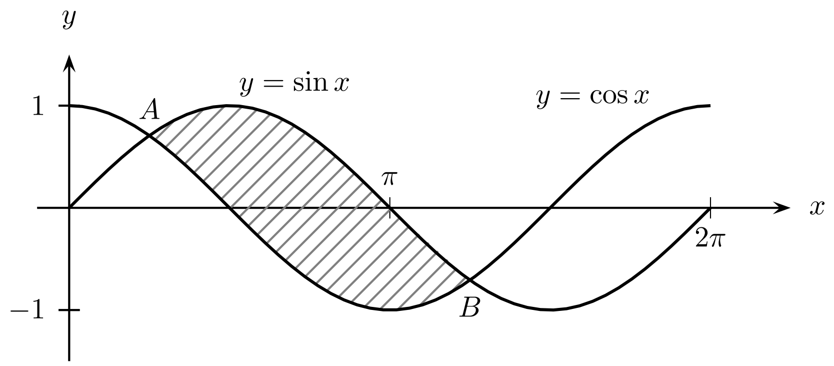
i. How many litres of water were in the tank when the leakage commenced? **1**

ii. At what rate is the water leaking after 5 hours? Answer correct to 1 decimal place. **2**

iii. How many litres will eventually be in the tank after a long period of time? **1**

iv. If the farmer realises the tank is leaking when the volume of water remaining is 6 000 L, how long did it take him to realise there was a hole in the tank? Answer correct to the nearest minute. **2**

(b) The diagram shows the graphs $y = \sin x$ and $y = \cos x$, $0 \le x \le 2\pi$. The graphs intersect at A and B.

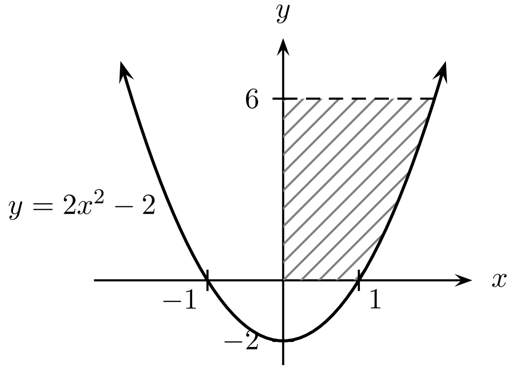


i. Show that A has coordinates $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$ and find the coordinates of B. **3**

ii. Find the area enclosed by the two graphs. **3**

Question 8 (12 Marks) Commence a NEW page. **Marks**

(a) The diagram shows the region bounded by the curve $y = 2x^2 - 2$, the line $y = 6$ and the x and y axes. **3**



Find the volume of the solid of revolution formed when the region is rotated about the y axis.

(b) Kevin plays computer games competitively. From past experience, Kevin has a 0.8 chance of winning a game of *Sawcraft* and a 0.6 chance of winning *CounterStrife*. During a LAN party he plays two games of *Sawcraft* and one of *CounterStrife*.

What is the probability that he will win:

i. all 3 games? **1**

ii. No games? **2**

iii. At least 1 game? **2**

(c) For the quadratic equation $x^2 + (p - 3)x - (2p + 1) = 0$,

i. Show that the discriminant is $\Delta = p^2 + 2p + 13$. **2**

ii. Hence or otherwise, show that the quadratic equation $x^2+(p-3)x-(2p+1) = 0$ will always have real, distinct roots for real valued p. **2**

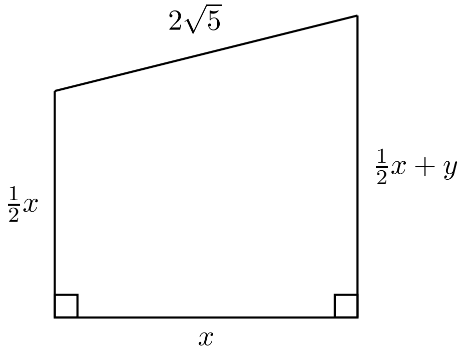
Question 9 (12 Marks) Commence a NEW page. **Marks**

(a) Consider the geometric series $1 + \frac{4}{3}\sin^2 x + \frac{16}{9}\sin^4 x + \frac{64}{27}\sin^6 x + \cdots$.

i. When the limiting sum exists, find its value in simplest form. **2**

ii. For what values of x in the interval $0 < x < \frac{\pi}{2}$ does the limiting sum of this series exist? **2**

(b) The diagram below represents (in metres) the dimensions of a small garden.



i. Show that $y = (20 - x^2)^{\frac{1}{2}}$. **2**

ii. Write an expression, in terms of x, for the perimeter P (in metres) of the garden, and find a value of x for which **4**

$$\frac{dP}{dx} = 0$$

iii. Establish whether this value of x gives a minimum or maximum value of P and find that value of P. **2**

Question 10 (12 Marks) Commence a NEW page. **Marks**

(a) A city has a growing population at a rate proportional to the current population, that is

$$\frac{dP}{dt} = kP$$

i. Verify that $P(t) = P_0 e^{kt}$, $t > 0$ is a solution of the equation. **1**

ii. If the population on 1 January 2006, which is $t = 1$, was 147 200 and on 1 January 2007 (when $t = 2$) was 154 800, find the initial population and the value of k. Round your answer down to the nearest whole number. **2**

iii. Find the population on 1 January 2009. **1**

iv. Find the time it will take for the population to double. **2**

(b) A car dealership has a car for sale for the cash price of \$20 000. It can also be purchased on terms over 3 years. The first 6 months are interest free. Subsequently, interest is charged at 12% per annum, calculated monthly. Repayments are to be made in equal monthly instalments at the end of the first month.

A customer purchases the car on these terms and agrees to monthly repayments of $\$M$ per month. Let $\$A_n$ be the amount owing at the end of the n-th month.

i. Find an expression for A_6. **1**

ii. Show that $A_8 = (20\,000 - 6M)\,1.01^2 - M(1 + 1.01)$. **2**

iii. Find an expression for A_{36}. **1**

iv. Find the value of M. **2**

End of paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \qquad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \ x > 0$

Suggested marking scheme

Question 1

(a) (2 marks)

✓ [1] for correct value.
✓ [1] for 3 significant figures.

$$\left(\frac{1}{e^{2.5}} - 1\right) = 0.843 \text{ (3 s.f.)}$$

(b) (2 marks)

✓ [1] for multiplying by the fraction with appropriate conjugate surd.
✓ [1] for final answer.

$$\frac{\sqrt{2}}{1+\sqrt{5}} \times \tfrac{1-\sqrt{5}}{1-\sqrt{5}} = -\frac{\sqrt{2}-\sqrt{10}}{4} = \frac{\sqrt{10}-\sqrt{2}}{4}$$

(c) (2 marks)

✓ [1] for correct usage of chain rule.
✓ [1] for final answer.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 3(4x+1)^2 \times 4\\ &= 12(4x+1)^2\end{aligned}$$

$$\begin{array}{|ll} y = u^3 & u = 4x+1\\ y' = 3u^2 & u' = 4\end{array}$$

(d) (2 marks)

✓ [1] for correctly factorising xy.
✓ [1] for correctly factorising cubic.

$$\begin{aligned}x^4y - xy^4 &= xy(x^3 - y^3)\\ &= xy(x-y)(x^2+xy+y^2)\end{aligned}$$

(e) i. (2 marks)

✓ [1] for identifying $32 = 2^5$.
✓ [1] for correct solution.

$$\begin{aligned}2^{2x-3} &= 32 = 2^5\\ 2x - 3 &= 5\\ 2x = 8 &\Rightarrow x = 4\end{aligned}$$

ii. (2 marks)

✓ [1] for correctly factorising quadratic.
✓ [1] for $x = -1$, 2.

$$\begin{aligned}x^2 - x &= 2\\ x^2 - x - 2 &= 0\\ (x-2)(x+1) &= 0\\ x &= -1, 2\end{aligned}$$

Question 2

(a) i. (1 mark)

$$2x + 3y + 6 = 0$$

When $y = 0$, $2x + 6 = 0$

$$2x = -6 \Rightarrow x = -3$$

ii. (1 mark)

$$2x + 3y + 6 = 0$$

When $x = 0$, $3y + 6 = 0$

$$3y = -6 \Rightarrow y = -2$$

iii. (2 marks)

✓ [1] for using midpoint formula.
✓ [1] for final answer $C(3, -4)$.

$$(0, -2) = \left(\frac{x_c + (-3)}{2}, \frac{y_c + 0}{2}\right)$$

$$\begin{array}{c|c} \dfrac{x_c - 3}{2} = 0 & \dfrac{y_c}{2} = -2\\ x_c = 3 & y_c = -4\end{array}$$

$$\therefore C(3, -4)$$

iv. (2 marks)

✓ [1] for correct gradient of k.
✓ [1] for correct y intercept of k.

$$\begin{aligned}m_\ell = -\frac{2}{3} &\Rightarrow m_k = \frac{3}{2} \text{ as } \ell \perp k\\ y &= \frac{3}{2}x + b\Big|_{\substack{x=3\\y=-4}}\\ -4 &= \frac{3}{2} \times 3 + b\\ \therefore b &= -4 - \frac{9}{2} = -\frac{17}{2}\\ \therefore y &= \frac{3}{2}x - \frac{17}{2} \Rightarrow 3x - 2y - 17 = 0\end{aligned}$$

v. (3 marks)

✓ [1] for each correct inequality.

$$\begin{cases} y \geq -\frac{2}{3}x - 2 \\ y \leq 0 \\ y \geq \frac{3}{2}x - \frac{17}{2} \end{cases} \qquad \begin{cases} 2x + 3y + 6 \geq 0 \\ y \leq 0 \\ 3x - 2y - 17 \leq 0 \end{cases}$$

(b) (3 marks)

✓ [1] for application of product rule.
✓ [1] for finding function value at $x = e$.
✓ [1] for final answer.

$$y = x^2 \ln x$$
$$u = x^2 \quad v = \ln x$$
$$u' = 2x \quad v' = \tfrac{1}{x}$$
$$\begin{aligned} \frac{dy}{dx} &= uv' + vu' = x^2 \cdot \tfrac{1}{x} + 2x \cdot \ln x \\ &= x + 2x \ln x \Big|_{x=e} \\ &= e + 2e \ln e = 3e \end{aligned}$$

The function value at $x = e$ is

$$y = x^2 \ln x \Big|_{x=e} = e^2$$

Substituting (e, e^2) into equation of the tangent,

$$\begin{aligned} \therefore y &= 3ex + b \Big|_{\substack{x=e \\ y=e^2}} \\ e^2 &= 3e^2 + b \\ b &= -2e^2 \\ \therefore y &= 3ex - 2e^2 \end{aligned}$$

Question 3

(a) i. (1 mark)

$$(x-2)^2 = 4 \times 2(y+1)$$
$$\therefore a = 2$$

ii. (1 mark)

$$V(2,-1) \quad a = 2$$
$$\therefore F(2, -1 + a) = F(2, 1)$$

iii. (1 mark)

$$\therefore y = -1 - a = -3$$

(b) i. (2 marks)

✓ [1] for correct differentiation of each term.

$$\frac{d}{dx}\left(2x^3 - x^{-1}\right) = 6x^2 + x^{-2}$$

ii. (1 mark)

✓ [1] for correct application of product or quotient rule
✓ [1] for correct final answer

$$\begin{aligned} \frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{e^{2x}\cos x - 2e^{2x}\sin x}{(e^{2x})^2} \\ &= \frac{\cancel{e^{2x}}(\cos x - 2\sin x)}{\cancel{e^{2x}} \cdot e^{2x}} \\ &= \frac{\cos x - 2\sin x}{e^{2x}} \end{aligned} \quad \begin{array}{|ll} u = \sin x & v = e^{2x} \\ u' = \cos x & v' = 2e^{2x} \end{array}$$

Alternatively, apply the product rule to $y = e^{-2x}\sin x$ to obtain

$$y' = e^{-2x}(\cos x - 2\sin x)$$

(c) (2 marks)

✓ [1] for finding the primitive.
✓ [1] for correct evaluation of limits.

$$\begin{aligned} \int_1^e \left(x^2 + \frac{2}{x}\right) dx &= \frac{1}{3}x^3 + 2\ln x \Big|_1^e \\ &= \frac{1}{3}\left(e^3 - 1\right) + 2(\cancel{\ln e}^{\,1} - \cancel{\ln 1}) \\ &= \frac{1}{3}e^3 + \frac{5}{3} \end{aligned}$$

(d) (3 marks)

✓ [1] recollection of Simpson's Rule
✓ [1] substitution of pronumerals.
✓ [1] evaluation.

$$\begin{aligned} A &\approx \frac{h}{3}\left(y_1 + 4\sum y_{\text{even}} + 2\sum y_{\text{odd}} + y_\ell\right) \\ &= \frac{\frac{1}{2}}{3}\left(12 + \cancel{4(8+3)}^{\,44} + 0 + 5\right) \\ &= \frac{1}{6}(17 + 44) = \frac{61}{6} \end{aligned}$$

Question 4

(a) i. (1 mark)

$$\frac{T_2}{T_1} = \frac{2.7}{3.0} = 0.9 \qquad \frac{T_3}{T_2} = \frac{2.43}{2.70} = 0.9$$

ii. (2 marks)

✓ [1] for substitution of pronumerals.

✓ [1] for correct answer to nearest minute.

$$T_n = ar^{n-1}$$
$$T_5 = 3 \times 0.9^4 = 1.9683 = 1 \text{ h } 59 \text{ min}$$

iii. (2 marks)

✓ [1] for substitution of pronumerals.

✓ [1] for correct answer to nearest minute.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_5 = \frac{3 \times (0.9^5 - 1)}{0.9 - 1}$$
$$= 12.2853 = 12 \text{ h } 17 \text{ min}$$

iv. (2 marks)

✓ [1] for correct evaluation resulting in $n \approx 9.31$.

✓ [1] for correct answer to $\lceil n \rceil$.

$$T_n = \underset{\div 3}{1.25} \text{ h} = \underset{\div 3}{3} \times 0.9^{n-1}$$
$$0.9^{n-1} = \frac{1.25}{3}$$
$$(n-1)\log 0.9 = \log \frac{1.25}{3}$$
$$n = \frac{\log \frac{1.25}{3}}{\log 0.9} + 1 \approx 9.31$$

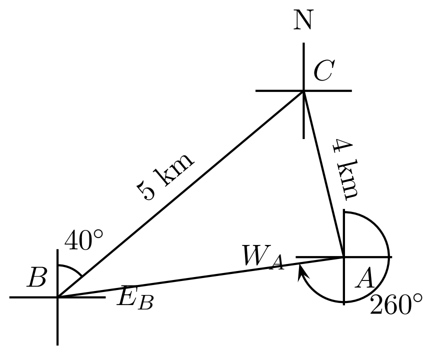
Michael must run for 10 weeks to improve on the previous record of 1 h 15 min.

(b) i. (2 marks)

✓ [1] for correct arithmetic.

✓ [1] for correct reasoning.

Any arithmetic/reasoning that is not acceptable will result in no marks awarded.



- $\angle W_A AB = 10°$ since the bearing of B from A is 260°.
- $\angle ABE_B = 10°$ (alt. $\angle$ on $\parallel$ lines)
- $\therefore CBA = 90° - 40° - 10° = 40°$. (complementary $\angle$)

ii. (2 marks)

✓ [1] for application of sine rule.

✓ [1] for final answer.

$$\frac{\overset{\times 5}{\sin 40°}}{4} = \frac{\overset{\times 5}{\sin \angle CAB}}{5}$$
$$\sin \angle CAB = \frac{5 \sin 40°}{4} \approx 0.803$$
$$\angle CAB = 53°28' = 53° \text{ (nearest °)}$$

iii. (1 mark)
The bearing of C from A is $53°28' + 260° = 313°28'$. Also accept 313°.

Question 5

(a) i. (1 mark)

$$f(x+h) = (x+h)^2 - (x+h)$$
$$= x^2 + 2hx + h^2 - x - h$$

ii. (3 marks)

✓ [1] for recollection of limit.

✓ [1] for substitution.

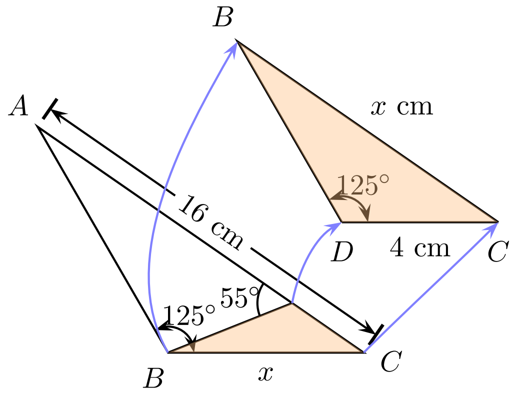
✓ [1] for final answer.

$$f(x+h) - f(x)$$
$$= (\cancel{x^2} + 2hx + h^2 - \cancel{x} - h) - (\cancel{x^2} - \cancel{x})$$
$$= 2xh + h^2 - h$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 - h}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 1)}{\cancel{h}} = 2x - 1$$

(b) i. (3 marks)

✓ [1] for each reason shown in summary.



In $\triangle ABC$ and $\triangle BDC$,

1. $\angle CDB = 180° - 55° = 125° = \angle ABC$ (supplementary $\angle$)
2. $\angle BCD$ common to $\triangle ABC$ and $\triangle BDC$.
3. $\therefore \angle DBC = \angle BAC$ (angle sum of $\triangle$) since two other pairs of angles are equal.

$\therefore \triangle ABC \parallel\!| \triangle BDC$ (AAA)

ii. (2 marks)

✓ [1] for relating corresponding sides in the same ratio.

✓ [1] for correctly evaluating x.

Since $\therefore$ $\triangle ABC \parallel\!| \triangle BDC$, all corresponding sides are in the same ratio, i.e.

$$\frac{x}{4} = \frac{16}{x}$$
$$x^2 = 64 \Rightarrow x = 8$$

(c) i. (1 mark)

$$2x^2 - 6x - 1 = 0$$
$$\alpha + \beta = -\frac{b}{a} = -\frac{-6}{2} = 3 \qquad (5.1)$$

ii. (1 mark)

$$\alpha\beta = \frac{c}{a} = -\frac{1}{2}$$

iii. (1 mark)

$$3\alpha - \alpha^2 = \alpha(3 - \alpha)$$

From (5.1), $\beta = 3 - \alpha$

$$\therefore \alpha(3 - \alpha) = \alpha\beta = -\frac{1}{2}$$

Question 6

(a) i. (2 marks)

✓ [1] for recollection of particle changing direction when $v = 0$.

✓ [1] for correct arithmetic and reasoning to obtain $t = 1$.

Particle changes direction when $v = 0$

$$\begin{aligned}
\frac{4}{t+1} &= 2t \\
4 &= 2t(t+1) \\
2t^2 + 2t - 4 &= 0 \\
t^2 + t - 2 &= 0 \\
(t+2)(t-1) &= 0 \\
t &= 1 \text{ since } t \geq 0
\end{aligned}$$

ii. (3 marks)

✓ [1] for applying absolute value to both terms of the distance.

✓ [1] for $d = |4\ln 2 - 1| + \left|4\ln\frac{3}{2} - 4\right|$.

✓ [1] for $d = 4\ln\frac{4}{3} + 3$ m.

✓ [Note:] If $d = \int_0^2 \frac{4}{t+1} - 2t\,dt$ is used, a maximum of [1] mark is awarded.

$$\begin{aligned}
d &= \left|\int_0^1 \frac{4}{t+1} - 2t\,dt\right| \\
&\quad + \left|\int_1^2 \frac{4}{t+1} - 2t\,dt\right| \\
&= \left|4\ln(t+1) - t^2\Big|_0^1\right| \\
&\quad + \left|4\ln(t+1) - t^2\Big|_1^2\right| \\
&= \left|4\left(\ln 2 - \cancel{\ln 1}\right) - \left(1^2 - \cancel{0^2}\right)\right| \\
&\quad + \left|4\left(\ln 3 - \ln 2\right) - \left(2^2 - 1^2\right)\right| \\
&= |4\ln 2 - 1| + \left|4\ln\tfrac{3}{2} - 3\right| \\
&= (4\ln 2 - 1) + \left(3 - 4\ln\tfrac{3}{2}\right) \\
&= 4\left(\ln 2 - \ln\tfrac{3}{2}\right) + 2 \\
&= 4\ln\tfrac{4}{3} + 2 \text{ m}
\end{aligned}$$

(b) i. (3 marks)

✓ [1] correct identification of $x = -1$, 3.

✓ [1] testing nature of stationary points.

✓ [1] finding coordinates & stating nature.

$$y = x^3 - 3x^2 - 9x + 1$$

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3)$$

$$= 3(x-3)(x+1)$$

Stationary pts. at $y' = 0$.

$$\therefore x = -1, 3$$

$$y = x^3 - 3x^2 - 9x + 1\Big|_{x=-2} = -1$$

$$y = x^3 - 3x^2 - 9x + 1\Big|_{x=-1} = 6$$

$$y = x^3 - 3x^2 - 9x + 1\Big|_{x=0} = 1$$

$$y = x^3 - 3x^2 - 9x + 1\Big|_{x=3} = -26$$

$$y = x^3 - 3x^2 - 9x + 1\Big|_{x=4} = -19$$

x	-2		-1	0	3		4
$\frac{dy}{dx}$		+	0	−	0	+	
y	-1	↗	6	↘	-26	↗	-19

$\therefore$ $(-1, 6)$ is a local maximum and $(3, -26)$ is a local minimum.

ii. (2 marks)

✓ [1] obtaining $y'' = 6x - 6$.

✓ [1] showing change in concavity when $x = 1$.

$$\frac{dy}{dx} = 3x^2 - 6x - 9 \Rightarrow \frac{d^2y}{dx^2} = 6x - 6$$

Pt. of inflexion when $y'' = 0$ & concavity change occurs, i.e.

$$6x - 6 = 0 \Rightarrow x = 1$$

$$y'' = 6x - 6\Big|_{x=0} < 0$$

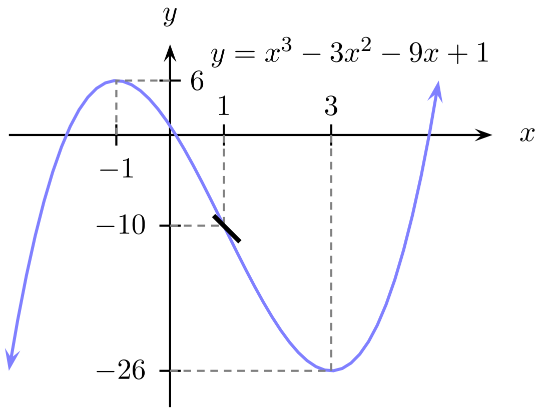
$$y'' = 6x - 6\Big|_{x=2} > 0$$

x	0	1	2
y''	⌢	0	⌣

iii. (2 marks)

✓ [1] shape of curve.

✓ [1] coords of stationary pts, pt. of inflexion.



Question 7

(a) i. (1 mark)

$$V(0) = 4\,000 + 10\,000e^0 = 14\,000 \text{ L}$$

ii. (2 marks)

✓ [1] for obtaining $V'(t) = -400e^{-0.04t}$.

✓ [1] for evaluating $V'(5) = -327.5$ L/h.

$$V(t) = 4\,000 + 10\,000e^{-0.04t}$$

$$V'(t) = -0.04 \times 10\,000e^{-0.04t}$$

$$= -400e^{-0.04t}$$

$$V'(5) = -327.5 \text{ L/h (1 d.p.)}$$

iii. (1 mark)

$$\lim_{t\to\infty} \left(4\,000 + \cancel{10\,000e^{-0.04t}}\right) = 4\,000 \text{ L}$$

iv. (2 marks)

✓ [1] for obtaining $e^{-0.04t} = \frac{1}{5}$.

✓ [1] for obtaining $t_1 = 40.23 = 40$ h 14 min.

Let t_1 be the time the farmer is aware of the leak.

$$V(t_1) = \underset{-4\,000}{6\,000} = \underset{-4\,000}{4\,000} + 10\,000e^{-0.04t}$$

$$\underset{\div 10\,000}{2\,000} = \underset{\div 10\,000}{10\,000}e^{-0.04t}$$

$$e^{-0.04t} = \tfrac{1}{5}$$

$$-0.04t = \ln\tfrac{1}{5}$$

$$t = \frac{\ln\frac{1}{5}}{-0.04} = 40.23\cdots \text{ h}$$

40 h 14 min have elapsed since the leak was discovered.

(b) i. (3 marks)

✓ [1] for solution of $\sin x = \cos x$.

✓ [2] for $A\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, $B\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$.

$$\begin{aligned}\sin x &= \cos x \\ \tan x &= 1\end{aligned} \quad \Rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\therefore A\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \qquad B\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$$

ii. (3 marks)

✓ [1] setting up integral.

✓ [1] for successfully finding primitive.

✓ [1] for $A = 2\sqrt{2}$.

$$\begin{aligned}A &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x\, dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos x\, dx \\ &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx \\ &= -\cos x - \sin x\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= -\left(\cos\tfrac{5\pi}{4} - \cos\tfrac{\pi}{4}\right) - \left(\sin\tfrac{5\pi}{4} - \sin\tfrac{\pi}{4}\right) \\ &= -\left(-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\right) - \left(-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\right) \\ &= \tfrac{2}{\sqrt{2}} + \tfrac{2}{\sqrt{2}} = 2\sqrt{2}\end{aligned}$$

Question 8

(a) (3 marks)

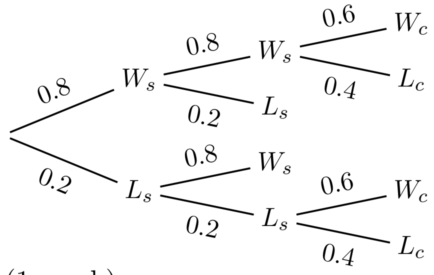
✓ [1] for changing the subject to x^2.

✓ [1] for setting up integral.

✓ [1] for solution $V = 15\pi$.

$$\begin{aligned}y = 2x^2 - 2 &\Rightarrow y + 2 = 2x^2 \\ x^2 &= \frac{y+2}{2} \\ V &= \pi \int x^2\, dy \\ &= \pi \int_0^6 \frac{y+2}{2}\, dy \\ &= \frac{\pi}{2}\left(\frac{1}{2}y^2 + 2y\right)\Big|_0^6 \\ &= \frac{\pi}{2}\left(\frac{1}{2}(6^2) + 2(6)\right) \quad [\tfrac{1}{2}(6^2) \to 18,\ 2(6) \to 12] \\ &= 15\pi \text{ units}^3\end{aligned}$$

(b) Tree diagram for this question:



i. (1 mark)

$$P(W_sW_sW_c) = (0.8)^2 \times (0.6) = 0.384$$

ii. (2 marks)

✓ [1] Find the complements of W_s, W_c:

$$\begin{aligned}P(L_s) &= 1 - P(\overline{W_s}) = 0.2 \\ P(L_c) &= 1 - P(\overline{W_c}) = 0.4\end{aligned}$$

✓ [1] correct evaluation of

$$P(L_sL_sL_c) = 0.016$$

$$P(L_sL_sL_c) = (0.2)^2 \times (0.4) = 0.016$$

iii. (2 marks)

✓ [1] Find the complement:

$$P(\text{win at least 1}) = 1 - P(L_sL_sL_c)$$

✓ [1] correct evaluation.

$$\begin{aligned}P(\text{win at least 1}) &= 1 - P(\text{win none}) \\ &= 1 - P(L_sL_sL_c) \\ &= 1 - 0.016 = 0.984\end{aligned}$$

(c) i. (2 marks)

$$a = 1 \quad b = (p-3) \quad c = -(2p+1)$$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (p-3)^2 - 4 \times 1 \times -(2p+1) \\ &= p^2 - 6p + 9 + 8p + 4 \\ &= p^2 + 2p + 13\end{aligned}$$

ii. (2 marks)

✓ [1] correctly evaluating $\Delta_\Delta < 0$.

✓ [1] logical reasoning.

$$\begin{aligned}\Delta_\Delta &= 2^2 - 4 \times 1 \times 13 = 4 - 42 < 0 \\ \therefore \Delta &= p^2 + 2p + 13 > 0\ \forall p\end{aligned}$$

Since $\Delta > 0$, therefore the quadratic always has real, distinct roots for real values of p.

Question 9

(a) i. (2 marks)

✓ [1] for correct identification $a = 1$, $r = \frac{4}{3}\sin^2 x$.

✓ [1] for correct substitution of a and r to the limiting sum $S = \dfrac{1}{1 - \frac{4}{3}\sin^2 x}$.

$$a = 1 \qquad r = \frac{4}{3}\sin^2 x$$

$$S = \frac{a}{1-r} = \frac{1}{1 - \frac{4}{3}\sin^2 x} \times \frac{3}{3}$$

$$= \frac{3}{3 - 4\sin^2 x}$$

ii. (2 marks)

✓ [1] for equating $0 < \left|\frac{4}{3}\sin^2 x\right| < 1$.

✓ [1] for finding $0 < x < \frac{\pi}{3}$.

$|r| < 1$ for limiting sum to exist.

$$\left|\frac{4}{3}\sin^2 x\right| < 1$$

$$0 < \frac{4}{3}\sin^2 x < 1$$

$$0 < \sin^2 x < \frac{3}{4}$$

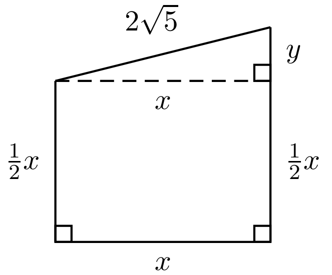
$$0 < \sin x < \frac{\sqrt{3}}{2}$$

$$0 < x < \frac{\pi}{3}$$

(b) i. (2 marks)

✓ [1] for using Pythagoras' Theorem

✓ [1] for showing required $y = (20 - x^2)^{\frac{1}{2}}$



$$x^2 + y^2 = \left(2\sqrt{5}\right)^2$$

$$x^2 + y^2 = 4 \times 5 = 20$$

$$y^2 = 20 - x^2 \Rightarrow y = (20 - x^2)^{\frac{1}{2}}$$

ii. (4 marks)

✓ [1] for obtaining

$$P = 2x + (20 - x^2)^{\frac{1}{2}} + 2\sqrt{5}$$

✓ [1] differentiating $P(x)$ correctly.

✓ [1] obtaining $x^2 = 16$.

✓ [1] concluding $x = 4$ as x is a length.

$$P = x + \tfrac{1}{2}x + \left(\tfrac{1}{2}x + y\right) + 2\sqrt{5}$$

$$= 2x + (20 - x^2)^{\frac{1}{2}} + 2\sqrt{5}$$

$$\frac{dP}{dx} = 2 + \frac{1}{\not{2}} \times (-\not{2}x) \times (20 - x^2)^{-\frac{1}{2}}$$

$$= 2 - \frac{x}{\sqrt{20 - x^2}} = 0$$

$$\frac{x^2}{20 - x^2} = 4$$

$$x^2 = 80 - 4x^2$$

$$\underset{\div 5}{5x^2} = \underset{\div 5}{80} \Rightarrow x^2 = 16$$

$\therefore x = 4$ since x is a length & $x > 0$

iii. (2 marks)

✓ [1] Checking $\frac{dP}{dx}$ within a neighbourhood of $x = 4$.

✓ [1] Evaluating $P(4) = 2\sqrt{5} + 10$.

$\dfrac{dP}{dx}$ exists when $x \geq 0$ and $x \leq \sqrt{20}$.

$$\frac{dP}{dx} = 2 - \left.\frac{x}{\sqrt{20 - x^2}}\right|_{x=3} = 1.095$$

$$\frac{dP}{dx} = 2 - \left.\frac{x}{\sqrt{20 - x^2}}\right|_{x=\sqrt{17}} = -0.38$$

x	3	4	$\sqrt{17}$
$\frac{dy}{dx}$	+	0	−
y	13.8 ↗	$2\sqrt{5} + 10$	↘ 14.5

$\therefore x = 4$ makes $P(4) = 2\sqrt{5} + 10$ a local maximum.

Question 10

(a) i. (1 mark)
Differentiating $P(t)$,

$$P'(t) = P_0 k e^{kt} = k \overbrace{P_0 e^{kt}}^{=P(t)} = kP$$

ii. (2 marks)
✓ [1] for obtaining $k = \ln \frac{387}{368}$.
✓ [1] for obtaining $P_0 = 139\,973$.

$$P(1) = 147\,200 = P_0 e^{k} \qquad (10.1)$$
$$P(2) = 154\,800 = P_0 e^{2k} \qquad (10.2)$$

(10.2) ÷ (10.1):

$$\frac{154\,800}{147\,200} = e^k$$
$$k = \ln \frac{387}{368} \qquad (10.3)$$

Substitute (10.3) to (10.1)

$$147\,200 = P_0 \exp\left(\ln \frac{387}{368}\right) = P_0 \times \frac{387}{368}$$
$$P_0 = \frac{147\,200 \times 368}{387} = 139\,973$$

iii. (1 mark)

$$P(4) = 139\,973 \exp\left(\ln \frac{387}{368} \times 4\right)$$
$$= 171\,197$$

iv. (2 marks)
✓ [1] for obtaining $\ln 2 = t \ln \frac{387}{368}$.
✓ [1] for obtaining $t \approx 14$ years.

$$P(t) = 2\cancel{P_0} = \cancel{P_0} \exp\left(\ln \frac{387}{368} \times t\right)$$
$$\ln 2 = t \ln \frac{387}{368}$$
$$t = \frac{\ln 2}{\ln \frac{387}{368}}$$
$$= 13.7688\cdots$$
$$= 14 \text{ years}$$

(b) i. (1 mark)
Since no interest is applied in the first 6 mths, then the amount owing will be

$$A_1 = 20\,000 - M$$
$$A_2 = 20\,000 - 2M$$
$$\vdots$$
$$A_6 = 20\,000 - 6M$$

ii. (2 marks)
✓ [1] for correctly finding A_7.
✓ [1] for correctly finding A_8.

After the 6th month, 12% p.a. $= 0.01$ p.m. interest is applied to A_6.

$$A_7 = A_6 \times 1.01 - M$$
$$= (20\,000 - 6M) \times 1.01 - M$$
$$A_8 = A_7 \times 1.01 - M$$
$$= \big((20\,000 - 6M) \times 1.01 - M\big) \times 1.01 - M$$
$$= (20\,000 - 6M) \times 1.01^2 - 1.01M - M$$
$$= (20\,000 - 6M) \times 1.01^2 - M(1 + 1.01)$$

iii. (1 mark)

$$A_9 = (20\,000 - 6M) \times 1.01^3 - M(1 + 1.01 + 1.01^2)$$
$$A_{36} = (20\,000 - 6M) \times 1.01^{30} - M(\underbrace{1 + 1.01 + 1.01^2 + \cdots + 1.01^{29}}_{30 \text{ terms}})$$

iv. (2 marks)
✓ [1] for finding $S_{30} = 100\left(1.01^{30} - 1\right)$.
✓ [1] for finding $M = \$628.78$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_{30} = \frac{1(1.01^{30} - 1)}{1.01 - 1} = 100\left(1.01^{30} - 1\right)$$

$A_{36} = 0$ as the loan is repaid and no amount is outstanding.

Letting $K = 1.01^{30}$,

$$(20\,000 - 6M) \times K = 100M\,(K - 1)$$
$$20\,000K - 6MK = 100M\,(K - 1)$$
$$20\,000K = 100M\,(K - 1) + 6MK$$
$$= M\,(100K - 100 + 6K)$$
$$= M\,(106K - 100)$$
$$\therefore M = \frac{20\,000 \times 1.01^{30}}{106 \times 1.01^{30} - 100} = 628.78$$

Their repayment is \$628.78 per month.